What is a k-graph?

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The plan

k-graphs and coloured graphs

Constructions

Dynamics and aperiodicity

Cubical sets and Topological realisations

C*-algebras

Section 1: k-graphs and coloured graphs

Definition

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> for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there are unique elements $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

Examples

- $E = (E^0, E^1, r, s)$ a directd graph. Recall $E^* = \{\text{finite paths}\}$ — a category under concatenation. Put $d(e_1 e_2 \cdots e_n) = n$. If $d(e_1 \cdots e_p) = m + n$, so $p = m + n$, have unique factorisation $(e_1 \ldots e_m)(e_{m+1} \cdots e_p)$. So (E^*, d) is a 1-graph.

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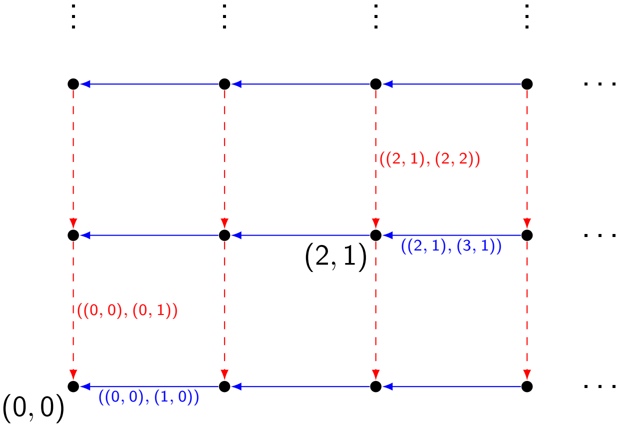
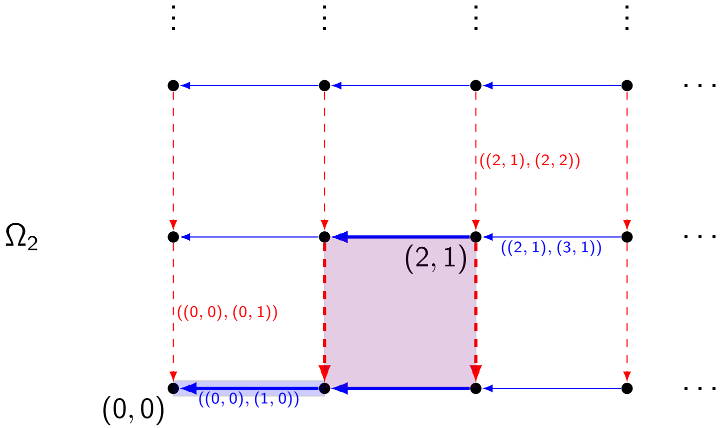
- $\mathbb{N}^k$ with $d = \mathrm{id} : \mathbb{N}^k \to \mathbb{N}^k$ is a k-graph.
- Define $\Omega_k := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$. Put $r(m, n) = (m, m)$, $s(m, n) = (n, n)$, $(m, n)(n, p) = (m, p)$, and $d(m, n) = n - m$.

Ω_2





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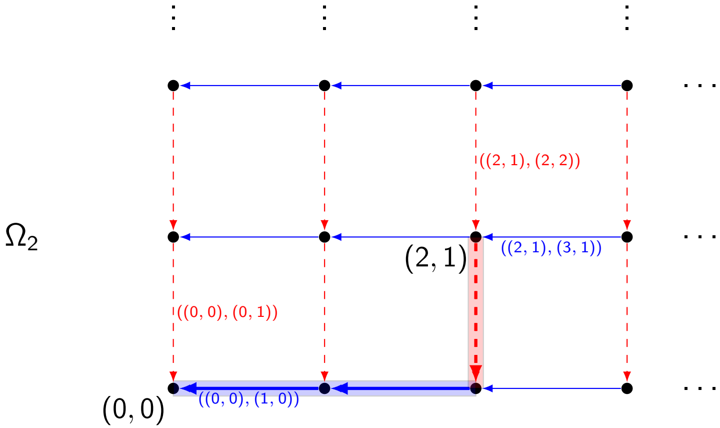

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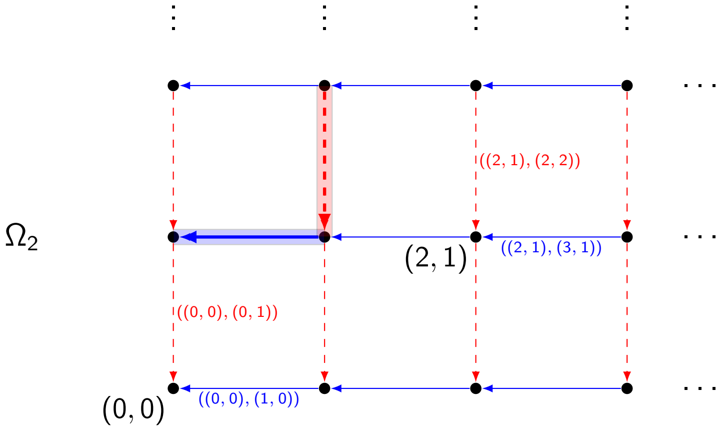

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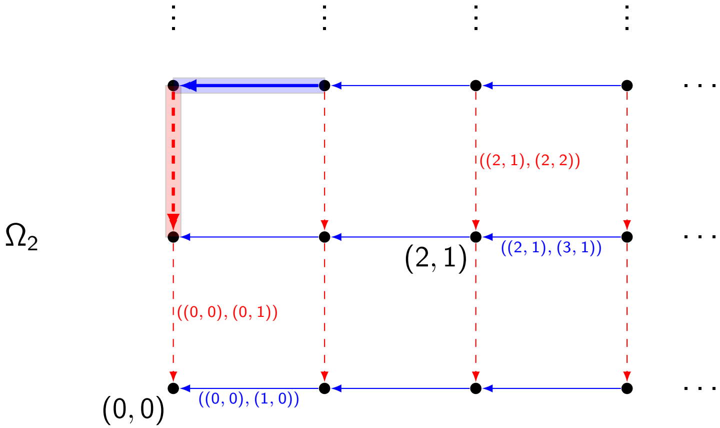

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Notation

- $\Lambda^n := d^{-1}(n)$.
- $r(\lambda) := \mathrm{id}_{\mathrm{cod}(\lambda)} \in \Lambda^0$ and $s(\lambda) := \mathrm{id}_{\mathrm{dom}(\lambda)} \in \Lambda^0$.
- For $E \subset \Lambda$ and $v \in \Lambda^0$, we write
  $vE := \{\lambda \in E : r(\lambda) = v\}$, and
  $Ev := \{\lambda \in E : s(\lambda) = v\}$.
- If $\lambda = \lambda'\lambda''\lambda'''$ with $d(\lambda') = m$ and $d(\lambda'') = d(\lambda) - n$, define $\lambda(m, n) := \lambda''$.

Elementary facts

Definition (Again)

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Cor. $\Lambda^{e_i} \, {}_s{*}_r \, \Lambda^{e_j} \cong \Lambda^{e_j} \, {}_s{*}_r \, \Lambda^{e_i}$.

Two-coloured graphs

Prop (Kumjian–Pask). Given graphs E_1, E_2 with common vertex set $E_1^0 = E_2^0$, and given an isomorphism

$$\theta_{12} : E_1 \;{}_s{*}_r\; E_2 \cong E_2 \;{}_s{*}_r\; E_1,$$

there is a unique 2-graph with $\Lambda^0 = E_1^0 = E_2^0$, $\Lambda^{e_i} = E_i^1$, and with $ef = f'e'$ whenever $\theta_{12}(e, f) = (f', e')$.

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Idea of proof: take path category of $(E_1^0, E_1^1 \cup E_2^1, r, s)$. Quotient by $\alpha ef \beta \sim \alpha f' e' \beta$ whenever $\theta_{12}(e, f) = (f', e')$.

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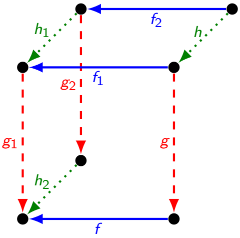

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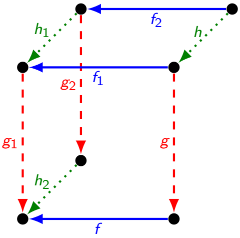

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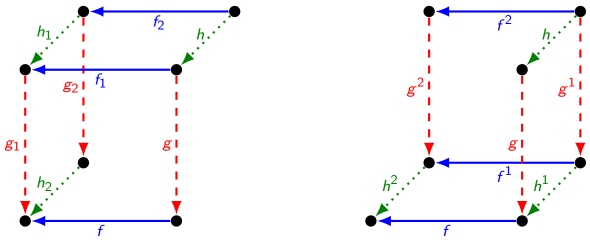

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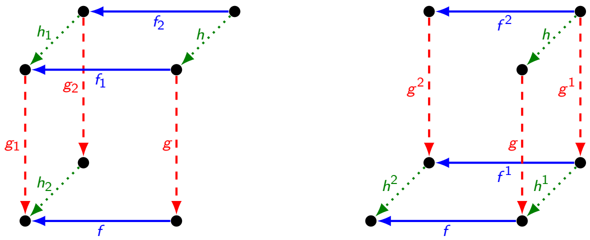

Associativity in Λ says $f_2 = f^2, g_2 = g^2, h_2 = h^2$.

An example

Is this associativity automatic whenever we have isomorphisms $E_i * E_j \cong E_j * E_i$ for $i \leq i, j \leq k$?

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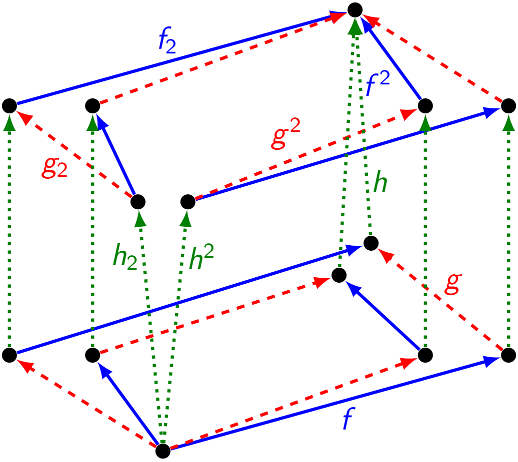
No. Example due to Spielberg:



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Definition

A *complete and associative collection of squares* for a $E_1, \ldots, E_k$ is a collection of isomorphisms $\theta_{ij} : E_i \,{}_s{*}_r\, E_j \cong E_j \,{}_s{*}_r\, E_i$ such that $\theta_{ji} = \theta_{ij}^{-1}$ and if we write $ef = f'e'$ when $\theta_{ij}(e, f) = (f', e')$, then if fgh is a tri-coloured path and

$$fg \sim g_1 f_1, \quad f_1 h \sim h_1 f_2, \quad g_1 h_1 \sim h_2 g_2,$$
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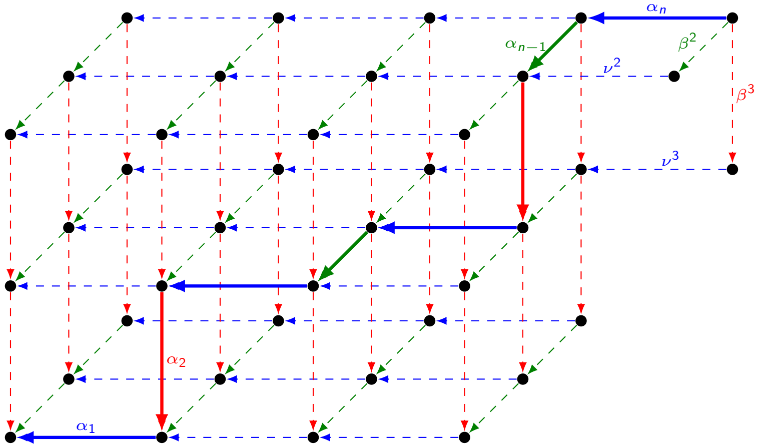
Theorem (Fowler–S, Hazelwood–Raeburn–S–Webster).
Every complete and associative collection of squares determines a k-graph and conversely.

Idea of proof

Inductively show that every path determines an entire commuting diagram.



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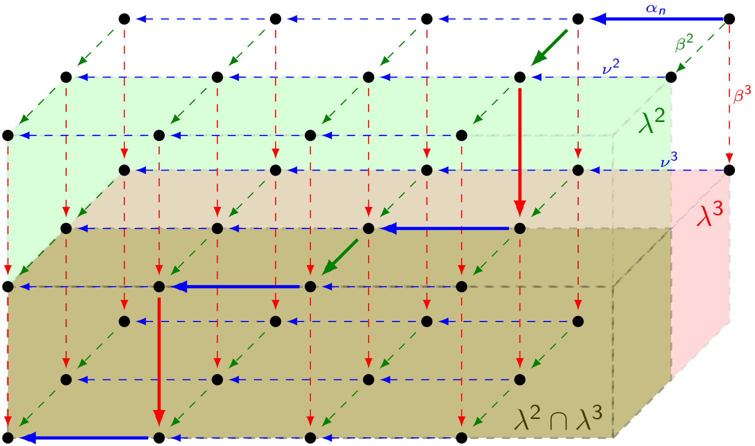

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Section 2: Constructions

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Cartesian products, pullbacks

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(Λ, d) a k-graph, $f : \mathbb{N}^l \to \mathbb{N}^d$ a semigroup homomorphism. Put $f^*\Lambda = \{(\lambda, m) \in \Lambda \times \mathbb{N}^l : f(m) = d(\lambda)\}$, with $d(\lambda, m) = m$, pointwise operations. This is an l-graph: the *pullback* l-graph.

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Λ a k-graph, $\mathbb{Z}^l$ acting on Λ by automorphisms. Put $\Lambda \rtimes \mathbb{Z}^l = \Lambda \times \mathbb{Z}$ with $r(\lambda, n) = r(\lambda)$, $s(\lambda, n) = \alpha_{-n}(s(\lambda))$. Put $(\lambda, m)(\mu, n) = (\lambda\alpha_m(\mu), m + n)$. This is a $(k + l)$-graph: the *crossed-product* $k + l$-graph.

Section 3: Dynamics and aperiodicity

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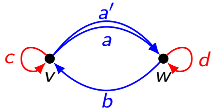

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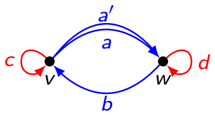

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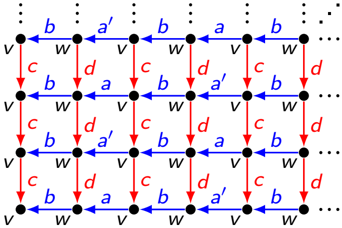

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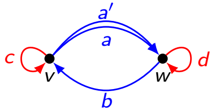

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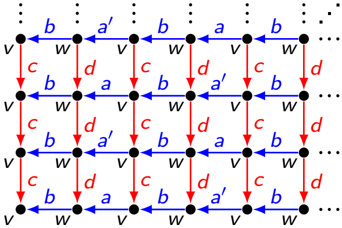

Then $\Lambda^\infty = \{\text{infinite paths}\}$ is a locally compact Hausdorff totally disconnected space with topology generated by $Z(\lambda) = \{x : x(0, d(\lambda)) = \lambda\}$.

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$f^*\Lambda$ rarely aperiodic: $f(m) = f(n)$ forces $\sigma^m(x) = \sigma^n(x)$ for all $x \in (f^*\Lambda)^\infty$.

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$\Lambda \times_d \mathbb{Z}^k$ is always aperiodic.

Shift maps

The semigroup $\mathbb{N}^k$ acts by local homeomorphisms of Λ^∞:

$$\sigma^p(x)(m,n) = x(m+p, n+p).$$

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$\Lambda \rtimes \mathbb{Z}^l$ aperiodic if $\{\Lambda$ aperiodic and $\mathbb{Z}^l \curvearrowright \Lambda^\infty$ free + transitive$\}$.

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Ledrappier shift

$$\Lambda^0:\quad \begin{array}{|c|c|}\cline{1-1} 0 & \multicolumn{1}{c}{} \\ \hline 0 & 0 \\ \hline \end{array}^{\,a} \qquad \begin{array}{|c|c|}\cline{1-1} 0 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline \end{array}^{\,b} \qquad \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 0 & 1 \\ \hline \end{array}^{\,c} \qquad \begin{array}{|c|c|}\cline{1-1} 1 & \multicolumn{1}{c}{} \\ \hline 1 & 0 \\ \hline \end{array}^{\,d}$$

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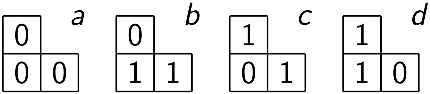
Λ^0: 

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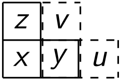
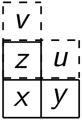
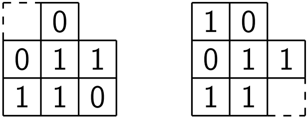
$\Lambda^{(1,0)}$: 

$\Lambda^{(0,1)}$: 

Factorisations: 

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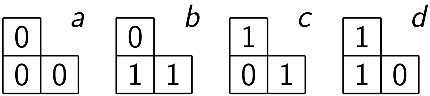

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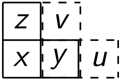

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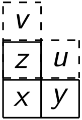

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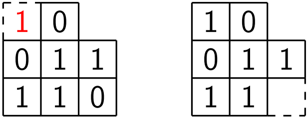

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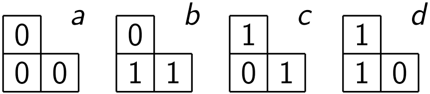
Λ^0: 

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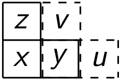
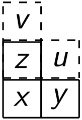
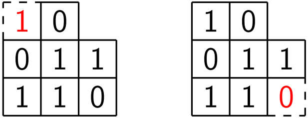
$\Lambda^{(1,0)}$: 

$\Lambda^{(0,1)}$: 

Factorisations: 

Section 4: Cubical sets and Topological realisations

A cubical set

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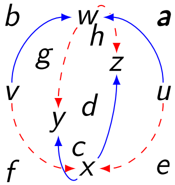

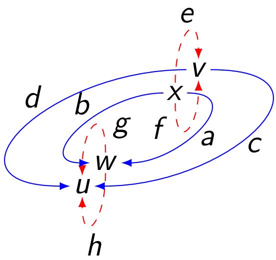

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Section 5: C*-algebras

The Cuntz–Krieger relations

If E is a row-finite directed graph, and $\{s_e, p_v\}$ a Cuntz–Krieger family, induction gives $s_\lambda^* s_\lambda = p_{s(\lambda)}$ and $p_v = \sum_{\lambda \in vE^n} s_\lambda s_\lambda^*$. Automatically have $s_\mu s_\nu = s_{\mu\nu}$ when μ, ν composable.

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Definition. Let Λ be a row-finite k-graph with no sources. A *Cuntz–Krieger Λ-family* is a collection $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries such that:

(CK1) $\{s_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;

(CK2) $s_\mu s_\nu = s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;

(CK3) $s_\mu^* s_\mu = s_{s(\mu)}$ for all $\mu \in \Lambda$; and

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Define $C^*(\Lambda)$ as the universal C^*-algebra generated by a Cuntz–Krieger family.

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Structure of $C^*(\Lambda)$

Theorem (Kumjian–Pask, Robertson–S). $C^*(\Lambda)$ is simple iff Λ is aperiodic and *cofinal* ($\forall v \in \Lambda^0, x \in \Lambda^\infty$, have $v\Lambda x(n) \neq \emptyset$ for large n).

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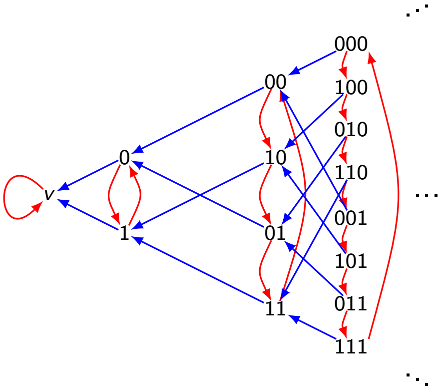

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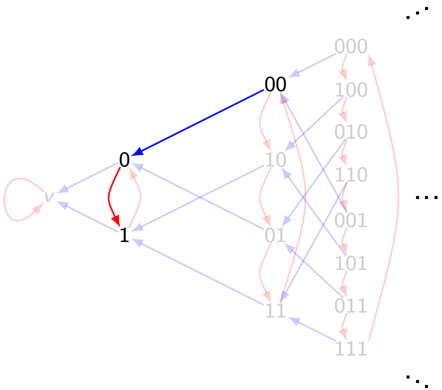

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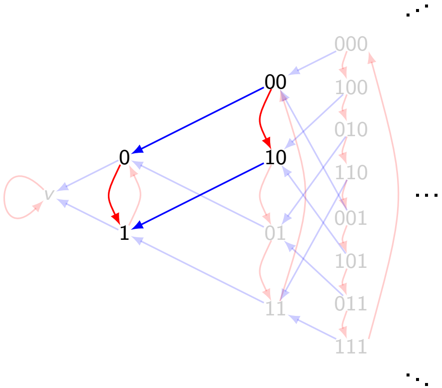

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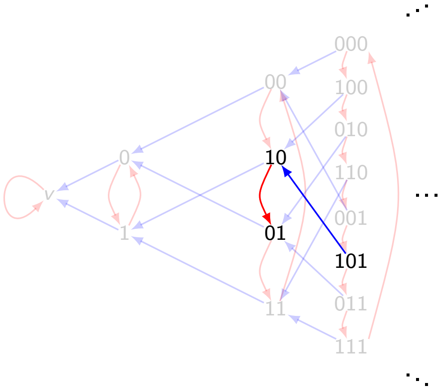

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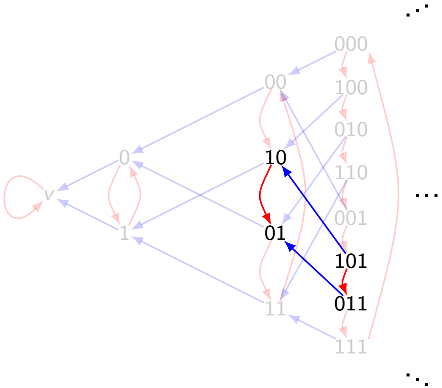

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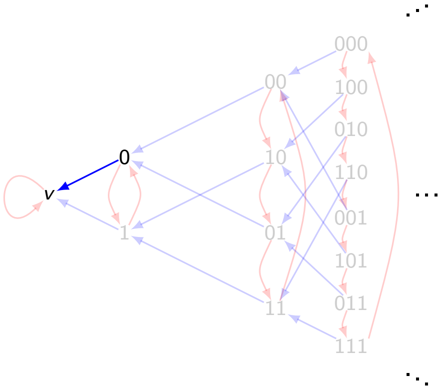

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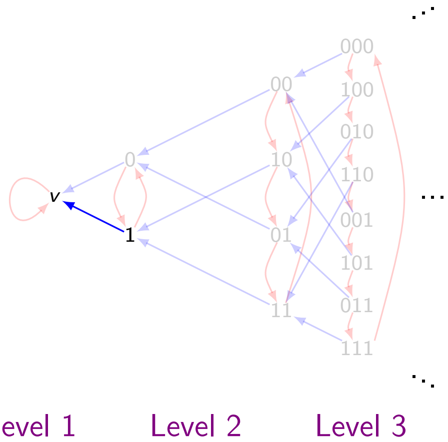

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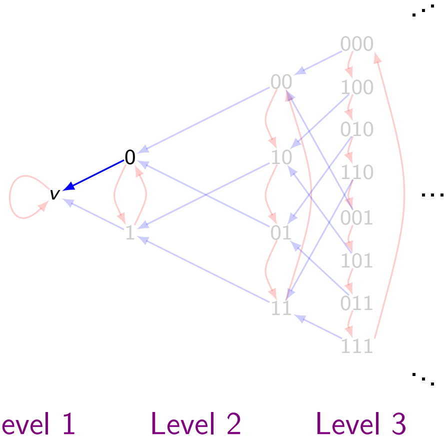

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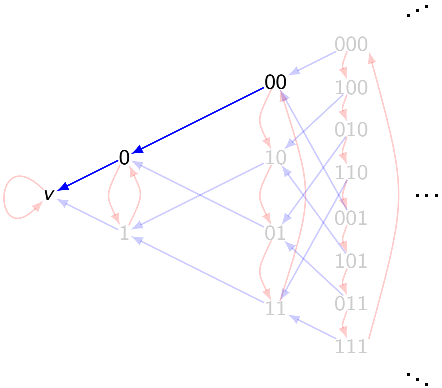

Level 1 Level 2 Level 3

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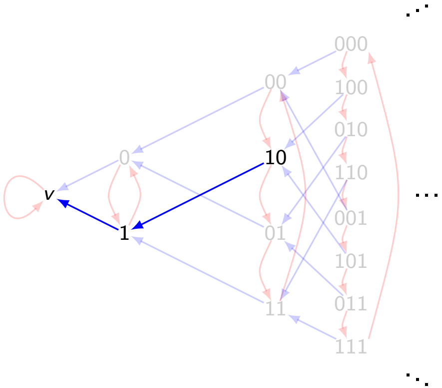

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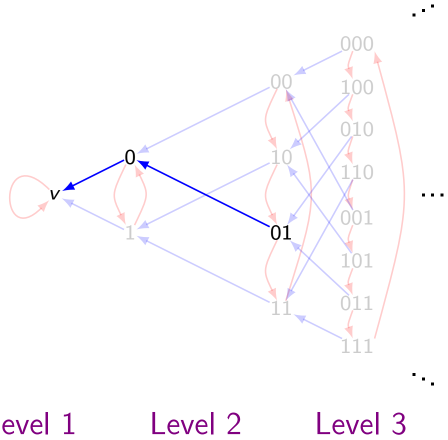

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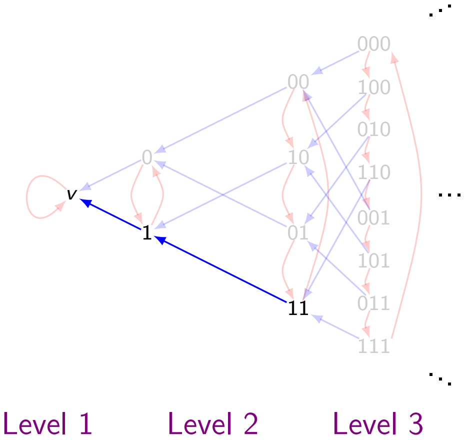

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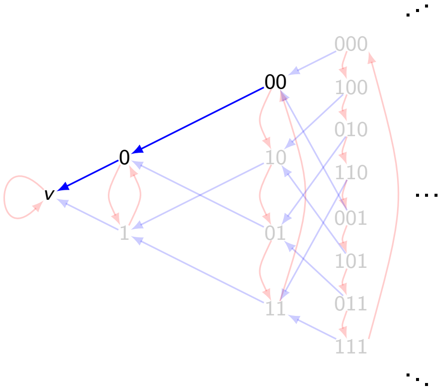

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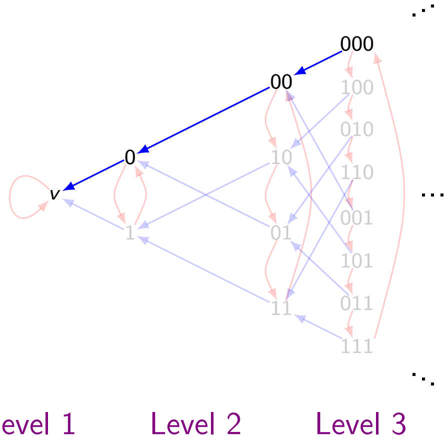

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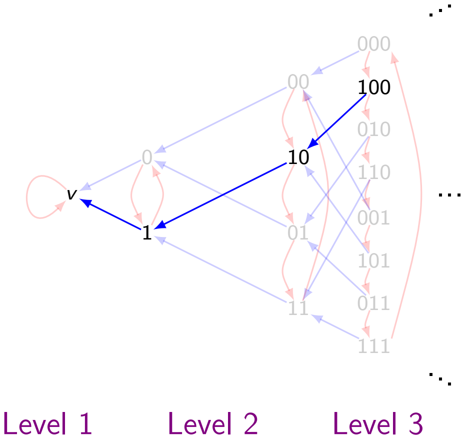

All

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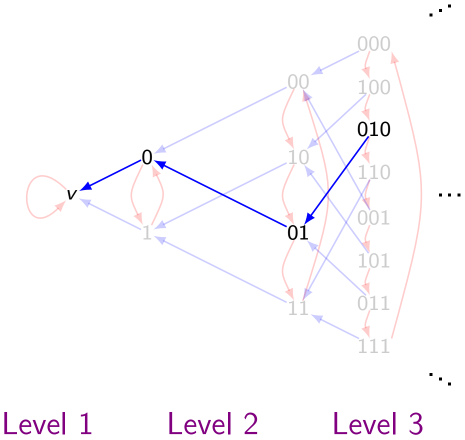
Example: No dichotomy



▸ All

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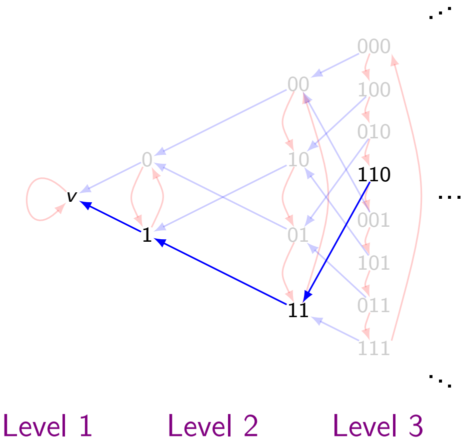

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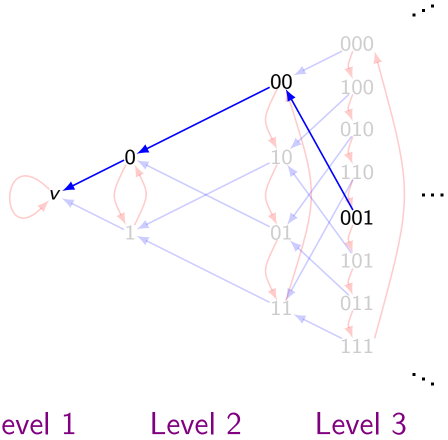

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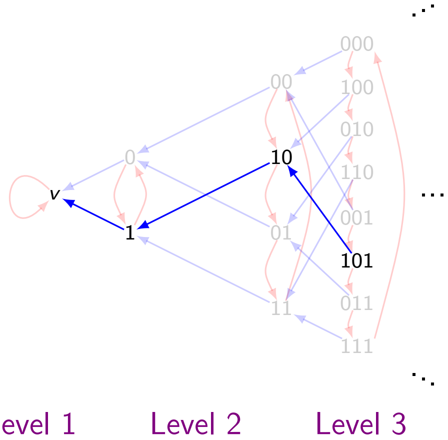

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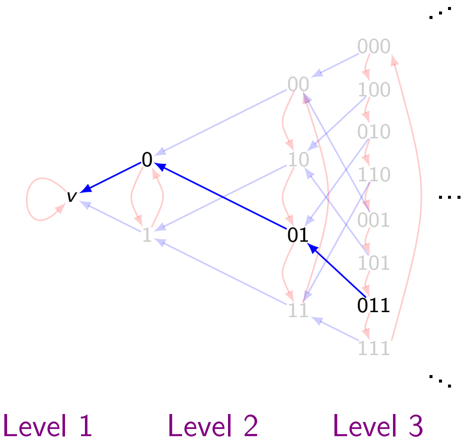

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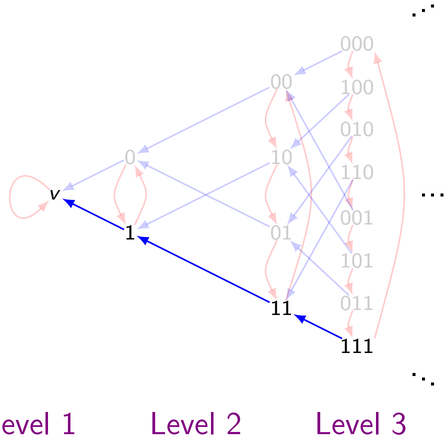

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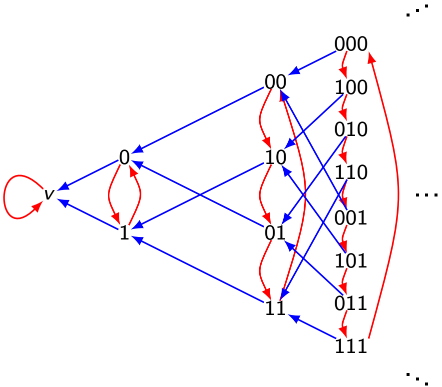

▸ All

$\mathbb{Z}$ acts on blue graph E by addition with carry. $\Lambda = E^* \rtimes \mathbb{Z}$.
On $v\Lambda^\infty$ this is the odometer. So $C^*(\Lambda) \sim$ Bunce–Deddens.